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Distributed control of flexible demand using proportional allocation mechanism in a smart grid: Game theoretic interaction and price of anarchy[☆]

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ABSTRACT

Demand side management leveraging flexibility of electric loads is a promising approach to reduce peak demand and to maintain supply–demand balance in future power grids with large scale renewable penetration. Distributed control is an important tool used for managing the demand side. In this paper, we develop a distributed method for coordinated control of flexible loads. In an intra-day setting, based on the supply schedule of thermal generators and predicted supply of renewable generators, a central control authority sets up price to maximize social welfare. This can be done using *proportional allocation mechanism* assuming that the consumers are price takers. If the consumers are *price anticipators*, the corresponding situation is modeled using non-cooperative game theory and existence of Nash equilibrium is established. Selfish behavior of agents in such a game theoretic set-up can reduce the efficiency of the overall system as compared to centralized control. In this paper, we are able to bound the efficiency loss for our game. The lower bound on the price of anarchy (PoA) for our game is found to be 0.75. For a non-cooperative game in a smart grid with flexible loads having different operating constraints and having utility functions that are non-separable with respect to time, this result is a new contribution to the demand response literature.

1. Introduction

Greenhouse gas emissions leading to climate change and long-term sustainability are some of the major reasons motivating adoption of renewable energy sources such as wind and solar into the electric energy system. Large-scale integration of wind and solar electric energy poses significant technological challenges. These energy sources are inherently uncertain (power generation not known in advance), intermittent (large fluctuations and ramps) and non-dispatchable (unable to follow a command). The term *variability* is used to represent these three characteristics [1]. This variability is a significant hurdle in integrating renewables at a large scale [2]. Balancing supply and demand, a critical requirement in power systems operations, becomes more challenging because of this inherent variability.

Increase of peak power demand as a result of economic and population growth, particularly in developing countries, is also a serious concern for the electric grid. Conventionally, the total installed power system generation, transmission and distribution capacity is dictated by the need to cater to the peak demand. Thus, power system infrastructure costs increase as peak demand increases. But this peak demand occurs only for a small fraction of time in a year [3].

A possible solution to address the increase of variability and/or peak power consumption is to deploy *demand side management* (DSM) or *demand response* (DR) program where consumers adjust their demand taking advantage of flexibility of their electric loads. Demand response program is considered to be a promising solution to integrate intermittent and uncertain renewable energy in today's electrical systems [4–7]. Albadi et al. [8] distinguish two types of DR programs: incentive based programs (IBP) and price based programs (PBP). The latter aim to reduce the demand by offering a higher price during peak periods and lower price during off-peak periods. These rates include among others, Time of Use (TOU) rate, Critical Peak Pricing (CPP), and Real Time Pricing (RTP). In [9], a study of the response of residential consumers to dynamic pricing of electricity has been accomplished by surveying fifteen experiments of dynamic prices. The conclusions of this study are that time-based rates is the most favorable case and including

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enabling technologies induce a drop in peak demand that may reach 27%–44%. Our approach to demand side management is a price based program maximizing social welfare of a group of flexible consumers and guaranteeing supply–demand balance. We aim to design a transparent market based approach and analyze the effect of behavior of the consumers on its efficiency.

1.1. Literature survey

In the traditional power systems, the generation is adjusted to meet the uncontrollable demand. But a paradigm shift in power system operations is underway where consumers will be incentivized to manage their demand by leveraging the flexibility of their loads such as electric vehicles (EV), plug-in hybrid vehicles (PHEV), air conditioning, heat pumps, water heaters, pool pumps, washers, dryers, refrigerators, etc. The concepts of DSM or DR in power systems operations utilize the potential benefits that can accrue from exploiting this flexibility in electric loads. Different types of electric loads and their potential in contributing to electrical power demand management have been analyzed in [10–12]. Development of new advanced sensing, computation, communication, and controls technologies will be needed to realize this potential. Datacenters are also considered as valuable resources for DSM programs to balance power supply and demand due to their huge energy consumption and their flexibility for temporal and spacial load distribution [13,14]. Stackelberg games are proposed to analyze DR incentive schemes for either geo-distributed [13] and colocation datacenters [14]. Datacenters and their tenants, in the case of colocation datacenters, respond to a price signal originated by the utility of the datacenter operator.

Modern electrical power systems are large-scale systems with thousands of consumers and loads. For these systems, it is difficult to have a single decision maker, because different agents can have different objectives. In this setting, distributed control can be used as a major tool to solve problems where a control authority sends a control signal (*e.g.* price of electricity) and consumers decide their consumption schedules according to some private utility function [15,16]. A strategy for assigning quantities in a distributed price-based framework is the proportional allocation mechanism. In this mechanism, the control authority calculates a price for all the consumers in such a way that the assigned quantity to each agent is proportional to the monetary value that the agent is willing to pay. This mechanism was applied by Kelly [17,18] in a traffic routing problem for a communication network. It has also been used to formulate a distributed EV charging control problem in [19] and to develop a coordinated charging policy for EVs, inspired by lottery scheduling in [20]. However, the mechanism has not been examined in a much broader system level setting in the smart grid context.

Consumer behavior plays a critical role in the implementation of demand response programs in distributed mode. The success of DSM programs requires active consumer participation. However, not every consumer behaves always as a rational agent. The Enron scandal in 2001 showed that market participants can manipulate prices [21] and the assumption of price taking consumers might not always be true. In [22,23], the role of consumer participation in the smart grid is analyzed using prospect theory to refine existing game-theoretic techniques.

Selfish behavior of agents in a non cooperative game leads to inefficiency with respect to the solution that maximizes system welfare. The loss of efficiency in a Cournot competition game can approach 100% [24]. Thus, it is crucial to design distributed control systems in such a way that the efficiency loss due to selfish behavior can be reduced and bounded. The term *price of anarchy* (PoA) has been coined as an index to measure this loss of efficiency. Deriving PoA bounds has become a topic of great research interest in different fields such as economics, communication and computer science [25–28]. Bounds on the PoA for various cost sharing games, congestion games and pay-off maximization games have been derived in [25,26]. A tight PoA bound, *i.e.* Pigou bound for routing games, is derived in [27]. A tight bound on the PoA in the problem of allocating a single infinitely divisible resource among multiple competing users is obtained in [28]. Finally, robust bounds on the PoA for *smooth games* are reported in [26].

In the smart grid scenario, though non-cooperative game theoretic methods have been used to model problems [29–32], the loss of efficiency by selfish behavior has not been widely investigated. The non-cooperative behavior of agents in a dynamic oligopoly smart grid market structure has been shown to lead to suboptimal outcome in comparison to cooperative behavior [31], but no bound on the PoA has been derived. The Nash equilibrium has been shown to be efficient in an infinite population game when the charging rates of all the EVs are equal [29], however these assumptions are rather impractical. The Nash equilibrium game has been shown to be efficient in a demand response problem with different consumers, but the utility functions of the consumers were ignored [32]. Considering wind variability, a game has been formulated among various power consumers in [30] where the PoA bound has been calculated for an example case. Game theoretic modeling with a Nash equilibrium is only useful when the performance at Nash equilibrium is shown to be good with respect to centralized control. In a demand response model with Cournot competition game setting, we were able to achieve a robust lower bound of 50% on the PoA [33].

1.2. Our work with key technical contribution

In this paper, we propose a distributed method for controlling the flexible demand of the consumers with intra-day forecasts. Flexible consumers are modeled as individually rational agents that maximize their net utilities while satisfying their consumption constraints. The consumers bid the monetary value they are willing to pay at each time interval and a central control authority computes a price signal based on proportional allocation mechanism. We analyze two scenarios: *price taking* and *price anticipating* consumers. In the first case, we prove that if the control authority uses a proportional allocation method, a competitive equilibrium that maximizes the system welfare is achieved. Moreover, this competitive equilibrium is equivalent to the optimal solution obtained by dual Lagrangian decomposition of the centralized control problem. However, since the consumers are aware that they can improve their individual welfare by anticipating the effect of their (and others') decisions on the system price. In this second case, the selfish behavior of the consumers is modeled using a non-cooperative Cournot game and we prove that a Nash equilibrium exists. The Nash equilibrium is inefficient and we are able to obtain a lower bound on the PoA of 75%, i.e., the loss of efficiency is no greater than 25%. We also show that efficiency can be improved by either recruiting consumers with similar utility functions, classifying them into groups of similar utility, or increasing the number of flexible consumers. Some preliminary results were reported in our previous paper [34], where only electrical vehicles were considered as flexible loads with utility functions that are separable in time. In this paper, we propose a more general model of various loads using utility functions that are not separable in time. There are many existing demand response models where the consumers bid their power demand and price is set up either by using marginal cost or some function of total demand and/or supply [15,16,29,30]. Unlike those models, in our case consumers bid the monetary value and the control authority uses that bid to generate a system price by using proportional allocation. This allocation mechanism produces a highly efficient game as described in [28]. The results

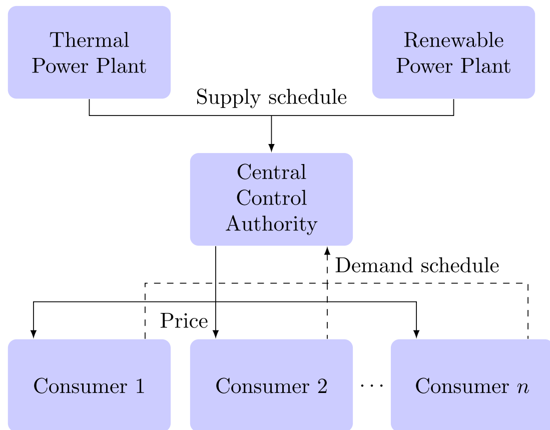

Fig. 1. System model.

in [28] are in a simplified setting where the users bid at a particular time and no operating constraints are present. Thus, they cannot be applied to our case. We formulate our model in a more general setting having multiple time intervals with the utility functions of the consumers not separable across time and also considering loads with linear inequality operating constraints. To the best of our knowledge, results on the use of proportional allocation rule and PoA analysis have not previously been reported in the demand response problem in this general context.

The remainder of this paper is organized as follows. In Section 2, we formulate the problem and introduce models for the consumers with flexible loads. In Section 3, we describe the centralized control problem that maximizes the system welfare. In Section 4, we use Lagrangian dual decomposition to convert the centralized control problem into a distributed control problem. In Section 5, we introduce the proportional allocation mechanism that also results in a distributed control problem. By considering the consumers as price takers, we prove that there is a price policy that produces a competitive equilibrium solution that maximizes the system welfare. The case of price anticipating consumers is formalized as a non-cooperative game in Section 6 and we prove the existence of a Nash equilibrium. In Section 7, we analyze the loss of efficiency for the selfish behavior of the consumers by obtaining a lower bound of the PoA and we study how to improve efficiency. A case study is presented in Section 8. Finally, the conclusions are discussed in Section 9.

2. Problem formulation

2.1. System model

Let us consider a residential area where electric power is supplied by thermal and renewable generators. The power consumption in the area is controlled by a central *control authority*. The control authority is considered as a non-profit agent whose role is twofold: maintaining supply–demand balance and maximizing system welfare. There are two types of residential consumers: *fixed consumers* and *flexible consumers*. Fixed consumers do not want to vary their consumption patterns, however flexible consumers are willing to adjust their consumption schedules in response to some signal from the authority. They can do that by harnessing the flexibility of different loads like EVs, PHEVs, air conditioners, heat pumps, washers, dryers, etc., whose consumption schedules and levels can be modified to a certain extent. A schematic model of the system is depicted in Fig. 1.

Let us consider a set $\mathcal{N}$ of flexible consumers, where each consumer is indexed by a natural number

$\mathcal{N} := \{k \in \mathbb{N} : k \le N\}$.

Each flexible consumer possesses a smart energy scheduling device with two-way communication capability. We assume that the supply is initially scheduled in a traditional day ahead market based on demand and renewable predictions. The time interval of interest $[t_0, t_f]$, corresponding to the intra-day horizon, is divided into T slots of length $\Delta t = (t_f - t_0)/T$. The set of time slots is

$\mathcal{T} := \{k \in \mathbb{N} : k \le T\}$,

and we consider the following variables at the tth time slot:
$q_i(t) \in \mathbb{R}_+$ is the power consumption of all the flexible loads of the ith flexible consumer. We assume that there is no power transfer from the consumers to the grid, *i.e.* $q_i(t) \ge 0$.
$c(t) \in \mathbb{R}_+$ is the total scheduled power generation of all the thermal power plants. This quantity is settled in the day-ahead market with some intra-day adjustments.
$\widehat{w}(t) \in \mathbb{R}_+$ is the estimate of the power generation of all the renewable sources.
$\widehat{n}(t) \in \mathbb{R}_+$ is the estimate of the total power consumption of all fixed loads of both fixed and flexible consumers.

We envision our method will be applied to an intra-day market 2–4 h of actual power consumption and T is assumed to be 8 h. In an intra-day time slot, during the operating day, the control authority obtains a forecast of renewable generation and balances the estimated demand with supply, *i.e.*,

$$c(t) + \widehat{w}(t) = \widehat{n}(t) + \sum_{i\in\mathcal{N}} q_i(t), \ t \in \mathcal{T}. \tag{1}$$

Let $v(t)$ denote the estimated net generation available for flexible demand at time slot $t \in \mathcal{T}$, *i.e.*

$$v(t) := c(t) + \widehat{w}(t) - \widehat{n}(t). \tag{2}$$

Using this new variable, the balance equation (1) simplifies to

$$v(t) = \sum_{i\in\mathcal{N}} q_i(t), \ t \in \mathcal{T}. \tag{3}$$

We assume that the net power supply is always sufficient to meet the fixed loads' demand, *i.e.* $v(t) > 0$ for any $t \in \mathcal{T}$. Then, the supply–demand balancing is accomplished by adjusting the power consumption of the flexible loads. There is always an inevitable mismatch between the estimated power generation and consumptions and their realized values. Ancillary services, such as load following and frequency regulation, are implemented to handle this real-time mismatch. However, the use of the intra-day flexible load control mechanism proposed here will reduce the need for ancillary services while making large scale renewable integration less burdensome as we will show next. Let us define the day-ahead and intra-day estimates of the net generation $v(t)$ as $v^d(t)$, $v^h(t)$ for all $t \in \mathcal{T}$, respectively. The corresponding power consumption schedules of the ith flexible consumer are denoted by $d_i^d(t)$ and $d_i^h(t)$ for all $i \in \mathcal{N}$, $t \in \mathcal{T}$, respectively. The real time values of the supply and demand are denoted by $v^r(t)$ and $d_i^r(t)$, respectively. The absolute value of the supply–demand imbalance at real time t is $\left|v^r(t) - \sum_{i\in\mathcal{N}} d_i^r(t)\right|$. This quantity is an indicator of the ancillary service that is needed. In absence of an intra-day demand control method, the supply and demand are jointly scheduled in the day-ahead market in such a way that $\sum_{i\in\mathcal{N}} d_i^d(t) = v^d(t)$, the real time consumptions $\sum_{i\in\mathcal{N}} d_i^r(t)$ are assigned using the day-ahead estimated available net power generation $v^d(t)$ and

$$\left|v^r(t) - \sum_{i\in\mathcal{N}} d_i^r(t)\right| = \left|v^r(t) - v^d(t)\right|. \tag{4}$$

However, if there exists an intra-day flexible demand control mechanism, $\sum_{i\in\mathcal{N}} d_i^h(t) = v^h(t)$, the real time consumptions are assigned using the intra-day estimated available net power generation and

$$\left|v^r(t) - \sum_{i\in\mathcal{N}} d_i^r(t)\right| = \left|v^r(t) - v^h(t)\right|. \tag{5}$$

Since the expected value of the absolute prediction error in the renewable energy production reduces with the prediction horizon, i.e.,

$$\mathbb{E}\left|v^r(t) - v^h(t)\right| \le \mathbb{E}\left|v^r(t) - v^d(t)\right|, \tag{6}$$

this implies from (4) and (5) that intra-day demand control reduces the average need for ancillary service in presence of renewables.

2.2. Model of flexible loads and consumers

Let $\mathbf{q}_i, \mathbf{v} \in \mathbb{R}_+^T$ denote vectors of dimension T that contain the flexible consumption of consumer $i \in \mathcal{N}$ and the net power generation available for flexible loads, respectively, for every $t\in\mathcal{T}$. For notational simplicity, and without loss of generality, we assume that each flexible consumer has only one flexible load. Let $U_i(\mathbf{q}_i) : \mathbb{R}^T \to \mathbb{R}$ denotes the utility function of a flexible consumer i in monetary unit. The utility function U_i is assumed to be a non-negative, concave and continuously differentiable function. U_i is also assumed to be a strictly increasing function, *i.e.* $\nabla U_i(\mathbf{q}_i) > 0$, where $\nabla U_i : \mathbb{R}^T \to \mathbb{R}^T$ denotes the gradient of U_i and $\nabla_t U_i$ is the tth element of the gradient.

Following [15], we consider three categories of the flexible loads: interruptible, thermostatically controlled and deferrable loads.

1. *Interruptible loads* are optional lighting, TV, video games, etc., for which the utility function at time t is only dependent upon the power consumption at time t. Its utility function is

$$U_i(\mathbf{q}_i) = \sum_{t\in\mathcal{T}} U_i(q_i(t)). \tag{7}$$

The load demand is constrained by $q_i^{\min}(t)$ and $q_i^{\max}(t)$, *i.e.*,

$$q_i^{\min}(t) \le q_i(t) \le q_i^{\max}(t). \tag{8}$$

If the load is supposed to not run at time t, then the bounds $q_i^{\min}(t), q_i^{\max}(t)$ are set to zero.

2. *Thermostatically controlled loads* are air conditioners, heaters and refrigerators, etc. At time t, the inside temperature of the load is $T_i^{\text{in}}(t)$, the outside temperature is $T_i^{\text{out}}(t)$ and the most comfortable inside temperature is $T_i^{\text{conf}}(t)$. The inside temperature evolves with time according to some dynamic difference equation [15]. The solution at time t is given by

$$T_i^{\text{in}}(t, \mathbf{q}_i) = (1-\alpha)^t T_i^{\text{in}}(0) + \sum_{\tau=1}^{t} (1-\alpha)^{t-\tau} \beta q_i(\tau) + \sum_{\tau=1}^{t} (1-\alpha)^{t-\tau} \alpha T_i^{\text{out}}(\tau), \tag{9}$$

where α and β are parameters that specify the thermal characteristics of the appliance and the environment in which it operates. These loads have a power consumption constraint like (8) and also a desired range of temperatures:

$$T_i^{\min}(t) \le T_i^{\text{in}}(t) \le T_i^{\max}(t). \tag{10}$$

The utility function has the form

$$U_i(\mathbf{q}_i) = \sum_{t\in\mathcal{T}} U_i(T_i^{\text{in}}(t, \mathbf{q}_i), T_i^{\text{conf}}(t)). \tag{11}$$

3. *Deferrable loads* include dishwasher, washer, dryer, PHEV, EVs, etc. The power consumption of these loads can be shifted to different times but these loads must consume a minimum energy within the time interval to complete some task. These loads have power constraints like (8) and also energy constraints as

$$Q_i^{\min} \le \sum_{t\in\mathcal{T}} q_i(t) \le Q_i^{\max}. \tag{12}$$

The utility is a function of the total energy consumed, *i.e.*,

$$U_i(\mathbf{q}_i) = U_i\left(\sum_{t\in\mathcal{T}} q_i(t)\right). \tag{13}$$

Thus for each consumer $i \in \mathcal{N}$, the utility function $U_i(\mathbf{q}_i)$ could be a combination of utility functions (7), (11) and (13) depending upon the type of loads being used for flexibility. Flexible load operating constraints can be expressed by the following linear inequalities

$$\mathbf{H}_i\mathbf{q}_i \le \mathbf{b}_i, \ i \in \mathcal{N}, \tag{14}$$

where $\mathbf{H}_i \in \mathbb{R}^{M\times T}$, $\mathbf{b}_i \in \mathbb{R}^M$ and M is the number of constraints. $\mathbf{H}_i$ and $\mathbf{b}_i$ can be written depending upon the flexible load type (8), (10) and (12). Let $\mathcal{S}_i$ denote the feasibility set of the consumptions for the consumer $i \in \mathcal{N}$, *i.e.*

$$\mathcal{S}_i = \{\mathbf{q}_i : \mathbf{H}_i\mathbf{q}_i \le \mathbf{b}_i\}. \tag{15}$$

We shall assume that this set is nonempty.

3. Centralized control

Let us consider an idealized scenario where the central control authority can dictate how much power each flexible consumer will receive at each time slot. The authority solves an optimization problem in order to maximize the social welfare, given by the aggregated utility of the flexible consumers, while satisfying supply–demand power balance constraint at each time. This optimization problem is called *the centralized control problem.*

Definition 1 (*The Centralized Control Problem*)**.** The centralized control problem is defined as follows:

$$\max_{\mathbf{q}_i} \left\{\sum_{i\in\mathcal{N}} U_i(\mathbf{q}_i) : \mathbf{q}_i \in \mathcal{S}\right\}, \tag{16}$$

where the search space

$$\mathcal{S} := \left\{\mathbf{q}_i \in \mathcal{S}_i : \mathbf{v} - \sum_{i\in\mathcal{N}} \mathbf{q}_i = \mathbf{0}\right\} \tag{17}$$

is assumed to be nonempty.

Since the objective function is concave and the search space is a compact convex set, the assumption that $\mathcal{S}$ is nonempty ensures that a maximum exist for this problem [35] and any solution that attains the maximum value is characterized by the Karush–Kuhn–Tucker (KKT) conditions. Let $\mathcal{L}^c$ denote the Lagrangian of this problem

$$\mathcal{L}^c = \sum_{i\in\mathcal{N}} U_i(\mathbf{q}_i) + \mathbf{p} \cdot \left(\mathbf{v} - \sum_{i\in\mathcal{N}} \mathbf{q}_i\right), \tag{18}$$

where $\{\mathbf{p} \in \mathbb{R}^T : i \in \mathcal{N}\}$ is the vector of Lagrange multipliers and $\cdot$ denotes the dot product of two vectors. The set of consumption vectors $\{\mathbf{q}_i^C : i \in \mathcal{N}\}$ solves the centralized control problem if there exists a vector of Lagrange multipliers $\mathbf{p}^C$ such that

$$\mathbf{q}_i^C \in \arg\max_{\mathbf{q}_i \in \mathcal{S}_i} (U_i(\mathbf{q}_i) - \mathbf{p}^C \cdot \mathbf{q}_i), \; i \in \mathcal{N}, \tag{19}$$

$$\mathbf{v} - \sum_{i \in \mathcal{N}} \mathbf{q}_i^C = \mathbf{0}. \tag{20}$$

This solution is referred to as the *centralized optimal solution* and is appealing because it is easy to formulate and it maximizes system welfare. However, it also has important drawbacks: the consumers may not want to disclose private information such as their utility functions and operating constraints, they may want to adjust their loads on their own, and the control authority may not have the computational capability to solve the optimization problem for a large number of residential consumers. As an alternative, we propose a distributed control approach based on Lagrangian dual decomposition [36].

4. Distributed dual decomposition method

Rearranging the terms, the Lagrangian of the centralized control problem given by Definition 1 can be written as follows

$$\mathcal{L}^c = \mathbf{p} \cdot \mathbf{v} + \sum_{i \in \mathcal{N}} (U_i(\mathbf{q}_i) - \mathbf{p} \cdot \mathbf{q}_i) \,. \tag{21}$$

The Lagrangian is clearly separable for each consumer, and so is the dual function g^c,

$$g^c(\mathbf{p}) = \mathbf{p} \cdot \mathbf{v} + \sum_{i \in \mathcal{N}} \sup_{\mathbf{q}_i \in \mathcal{S}_i} (U_i(\mathbf{q}_i) - \mathbf{p} \cdot \mathbf{q}_i) \,. \tag{22}$$

Since strong duality holds, the optimal values attained by the primal and dual problems are the same and we can recover a set of primal optimal points $\{\mathbf{q}_i^* : i \in \mathcal{N}\}$ from a set of dual optimal points $\{\mathbf{p}^* : i \in \mathcal{N}\}$ as

$$\mathbf{q}_i^* \in \arg\max_{\mathbf{q}_i \in \mathcal{S}_i} \left(U_i(\mathbf{q}_i) - \mathbf{p}^* \cdot \mathbf{q}_i\right) . \tag{23}$$

The dual function is always concave and efficient numerical algorithms can be developed to find dual optimal points. Let us assume that

$$\mathbf{q}_i^+ \in \arg\max_{\mathbf{q}_i \in \mathcal{S}_i} (U_i(\mathbf{q}_i) - \mathbf{p} \cdot \mathbf{q}_i) \,, \tag{24}$$

then we have that the residual of the equality constraint is a subgradient of the dual function

$$\partial g^c(\mathbf{p}) = \mathbf{v} - \sum_{i \in \mathcal{N}} \mathbf{q}_i^+, \tag{25}$$

and the dual function can be minimized using the following iterative method:

$$\mathbf{q}_i^{k+1} \in \arg\max_{\mathbf{q}_i \in \mathcal{S}_i} \left(U_i(\mathbf{q}_i) - \mathbf{p}^k \cdot \mathbf{q}_i\right) , \tag{26}$$

$$\mathbf{p}^{k+1} = \mathbf{p}^k - \alpha^k \left(\mathbf{v} - \sum_{i \in \mathcal{N}} \mathbf{q}_i\right) . \tag{27}$$

If the step size α_k is a nonsummable diminishing sequence, *i.e.* $\lim_{k\to\infty}\alpha_k = 0$, but $\sum_{k=1}^{\infty}\alpha_k = \infty$, then $\lim_{k\to\infty}(\mathbf{v} - \sum_{i\in\mathcal{N}}\mathbf{q}_i^k)$ converges to zero [37,38].

The dual vector variable $\mathbf{p}$ can be interpreted as a price vector and therefore each consumer is maximizing her net utility (23). This motivates the following subgradient algorithm:

Algorithm 1 (*Distributed Algorithm Based on Dual Decomposition*)**.**

1. The control authority broadcasts the available net generation $\mathbf{v}$ and a price vector $\mathbf{p}$ to all the consumers.
2. Each consumer bids the consumption vector $\mathbf{q}_i$ that maximizes her net utility.
3. The control authority computes the residual $\mathbf{r} = \mathbf{v} - \sum_{i\in\mathcal{N}} \mathbf{q}_i$ and updates the price vector using a subgradient algorithm with the residual $\mathbf{r}$ as a subgradient and a right step size α_k. The control authority then broadcasts the new value of price to all the consumers.
4. Return to step 2 until the residual satisfies $|r| < \rho$, where $\rho > 0$ is a sufficiently small number.

The advantage of a distributed algorithm is that it is highly parallelizable, producing a drastic reduction in computational burden. Consider a system with a million consumers and 8 time slots. Then, the control authority has to solve a centralized maximization problem with 8 million variables and possibly a higher number of constraints. Moreover, the consumers should disclose private information such as their utilities and operating constraints. Alternatively, using a distributed algorithm each consumer solves a private and very small optimization problem with only 8 variables and submits the result. The control authority using the received information updates the prices by performing a subgradient iteration.

The subgradient algorithm is known to converge under an appropriate step size update, although this iterative convergence process is known to be slow. The convergence of the subgradient algorithm under several step size rules have been analyzed, *e.g.* constant step size, square summable but not summable, diminishing step size and nonsummable diminishing step length [39,40]. For any choice of the step size rule, the number of iterations to achieve a guaranteed relative accuracy of ϵ in the residual $|r|$ with respect to the initial value of the residual $|r_0|$ is proportional to $1/\epsilon^2$, *i.e.* the complexity of the subgradient algorithm is $O(1/\epsilon^2)$ [40]. In our case, $\epsilon = \rho|r_0|$ where ρ is the absolute desired accuracy. Moreover, this is an optimal result and cannot be improved [40]. However, since the computational load of each iteration is very low, because the consumers solve their corresponding optimization problems in parallel, the complete process can be accomplished in a few minutes, making the distributed algorithm a feasible and very efficient approach.

5. Distributed control using proportional allocation mechanism with price taking flexible consumers

We propose an alternative to the distributed mechanism based on dual decomposition presented in the previous section. The mechanism is called proportional allocation method as proposed by Kelly [17]. In this mechanism, each consumer bids the amount of money which she is willing to pay for the desired consumption. The control authority decides the price and assigns to each consumer a portion of the available resource that is directly proportional to her bid and inversely proportional to the price. When each consumer chooses a bid that maximizes her net utility, then there exists a choice of the price that will enable the consumers to maximize the system welfare. The consumers are assumed to be *price takers*, *i.e.* they do not anticipate the effect of their bids on the price.

Let $k_i(t)$ denote the amount of money the consumer $i \in \mathcal{N}$ is willing to pay for the consumption at time t independently of the price $p(t)$ and is called monetary value or expenditure of a consumer. The vector $\mathbf{k}_i \in \mathbb{R}_+^T$ collects the expenditures of consumer $i \in \mathcal{N}$ for every time slot $t \in \mathcal{T}$. Each consumer bids this vector to the control authority that computes a system price $\mathbf{p}$ in such a way that $\sum_{i\in\mathcal{N}} k_i(t)/p(t) = v(t)$ for all $t \in \mathcal{T}$.

Definition 2 (*The Proportional Allocation Mechanism*)**.** The proportional allocation of the energy consumption at time slot $t \in \mathcal{T}$ is given by:

$$q_i(t) = \frac{k_i(t)}{p(t)}, \quad i \in \mathcal{N}, \tag{28}$$

where $p(t) \neq 0$ is the price of electricity at time $t \in \mathcal{T}$, obtained by

$$p(t) = \frac{\sum_{i\in\mathcal{N}} k_i(t)}{v(t)}, \quad t \in \mathcal{T}. \tag{29}$$

Since $v(t)$ is always positive, the system price $p(t)$ is well defined for every time slot $t \in \mathcal{T}$ and guarantees:

$$v(t) = \sum_{i\in\mathcal{N}} q_i(t), \quad t \in \mathcal{T}. \tag{30}$$

Each consumer (flexible or fixed) is charged at the system price. Let the net utility of a consumer be defined as the total utility minus the expenditure. The flexible consumers maximize their own net utility function by a suitable selection of their consumptions $\mathbf{q}_i$. Let $\mathbf{p} \in \mathbb{R}^T$ denote the vector that collects the system prices for every time slot $t \in \mathcal{T}$ and $\mathbf{D}(\mathbf{p})$ be a diagonal square matrix whose main diagonal is given by vector $\mathbf{p}$. Then, the following distributed control problem can be defined.

Definition 3 (*The Distributed Control Problem with Price Takers*)**.** The distributed control problem for price takers is given by

$$\mathbf{k}_i^*(\mathbf{p}) \in \arg\max \left\{ U_i(\mathbf{D}^{-1}(\mathbf{p})\mathbf{k}_i) - \mathbf{1}\cdot\mathbf{k}_i : \mathbf{D}^{-1}(\mathbf{p})\mathbf{k}_i \in \mathcal{S}_i \right\}, \ i \in \mathcal{N}, \tag{31}$$

$$\mathbf{p} = \mathbf{D}^{-1}(\mathbf{v}) \sum_{i\in\mathcal{N}} \mathbf{k}_i^*(\mathbf{p}). \tag{32}$$

The solution of the distributed control problem with price takers is a *competitive equilibrium* that is defined as follows:

Definition 4 (*Competitive Equilibrium*)**.** The set $\{(\mathbf{k}_i^E, \mathbf{p}^E) : i \in \mathcal{N}\}$ is a *competitive equilibrium* if $\mathbf{k}_i^*(\mathbf{p}^E) = \mathbf{k}_i^E$ for $\mathbf{k}_i^*(\mathbf{p})$ given by Eq. (31) and the price vector $\mathbf{p}^E$ satisfies Eq. (32).

The competitive equilibrium always exists if the search space $\mathcal{S}$ (17) is nonempty, because a competitive equilibrium is equivalent to a solution of the centralized control problem.

Theorem 1. *The set* $\{(\mathbf{k}_i^E, \mathbf{p}^E) : i \in \mathcal{N}\}$ *is a competitive equilibrium if and only if the set of consumptions* $\{\mathbf{q}_i^E : i \in \mathcal{N}\}$ *where* $\mathbf{q}_i^E = \mathbf{D}^{-1}(\mathbf{p}^E)\mathbf{k}_i^E$ *is a solution to the centralized control problem.*

Proof. Since $(\mathbf{k}_i^E, \mathbf{p}_i^E)$ satisfies Eq. (31), by doing the change of variables $\mathbf{k}_i^E = \mathbf{D}(\mathbf{p}^E)\mathbf{q}_i^E$, we obtain

$$\mathbf{q}_i^E = \mathbf{D}^{-1}(\mathbf{p}^E)\mathbf{k}_i^E \in \arg\max \left\{ U_i(\mathbf{q}_i) - \mathbf{p}^E \cdot \mathbf{q}_i^E : \mathbf{q}_i \in \mathcal{S}_i \right\}, \ i \in \mathcal{N},$$

and Eq. (32) guarantees that $\mathbf{v} - \sum_{i\in\mathcal{N}} \mathbf{q}_i^E$. Then, $(\mathbf{k}_i^E, \mathbf{p}_i^E)$ is a competitive equilibrium if and only if $\{(\mathbf{q}_i^E, \mathbf{p}^E) : i \in \mathcal{N}\}$ is a solution to the centralized control problem (19) and (20). □

The following algorithm computes a competitive equilibrium solution:

Algorithm 2 (*Competitive Equilibrium Solution*)**.**

1. The control authority broadcasts the available net generation $\mathbf{v}$ and a price vector $\mathbf{p}$ to all the consumers.
2. Each consumer bids the monetary value vector $\mathbf{k}_i$ maximizing her net utility.
3. The control authority computes the residual $\mathbf{r} = \mathbf{v} - \mathbf{D}^{-1}(\mathbf{p}) \sum_{i\in\mathcal{N}} \mathbf{k}_i$ and updates the price vector using a subgradient algorithm with the residual $\mathbf{r}$ as a subgradient and a right step size α_k. The control authority then broadcasts the new value of price to all the consumers.
4. Each consumer updates her own monetary value vector $\mathbf{k}_i$ by solving the problem (31) with the new price vector and bids it.
5. Return to step 3 until the residual satisfies $|r| < \rho$, where $\rho > 0$ is a sufficiently small number.

The Kelly's proportional allocation mechanism was designed to compute the price vector and assign the consumptions in one step using Eq. (32), assuming that the consumers bid the optimal expenditures. However, the consumptions assigned using this price vector do not necessarily satisfy the operating constraints (14), and an iterative refinement process is needed. Step 3 in Algorithm 2 implements the same subgradient algorithm used for the distributed algorithm based on dual decomposition. Thus, Algorithm 2 converges to a competitive equilibrium and its complexity is $O(1/\epsilon^2)$. The optimization problem (31) that has to be solved by each consumer in Step 4 is convex and can be solved in the worst case in $O(1/\epsilon)$ iterations using a projected gradient algorithm. By using interior point methods, the rate of convergence can be improved to be superlinear [35,41].

6. Distributed control with price anticipating flexible consumers

The competitive equilibrium solution of distributed control with price taking consumers maximizes the system welfare. However, an individually rational consumer is not always interested in the system welfare, but in her own profit. Thus, we consider here the case where each consumer is aware that her consumption decisions may influence the price and take advantage of that. This means that the flexible consumers are modeled as price anticipators that try to account for the impact of their decisions on the system price $\mathbf{p}$ and adjust their decisions accordingly. By using the vectors of monetary values $\mathbf{k}_i$ as decision variables, where $k_i(t) = p(t)q(t)$ for $t \in \mathcal{T}$, the consumers can express the price vector $\mathbf{p}$ as a function of $\sum_{i\in\mathcal{N}} \mathbf{k}_i$. We assume that they know that $\mathbf{p}$ is decided by the formula $p(t) = \sum_{i\in\mathcal{N}} k_i(t)/v(t)$. Each consumer's monetary value depends on the sum of all the consumers' expenditure and the consumption assignment can be modeled as a non-cooperative Cournot game with complete information where consumers compete for their monetary values which in turn allows them to set the power consumptions. The players are the flexible consumers indexed by the set $\mathcal{N}$. The problem can be formulated in terms of only the monetary expenditures by eliminating the price and the consumptions variables. Let $\mathbf{k}_{-i} = \{\mathbf{k}_j : j \in \mathcal{N} \setminus \{i\}\}$ denote the collection of monetary value vectors of all flexible consumers other than the consumer $i \in \mathcal{N}$. Note that $\mathbf{p}$ and $\mathbf{q}_i$ can be expressed as functions of $\mathbf{k}_i$ as follows:

$$\mathbf{p}(\mathbf{k}_i, \mathbf{k}_{-i}) = \mathbf{D}^{-1}(\mathbf{v}) \sum_{j\in\mathcal{N}} \mathbf{k}_j,$$

$$\mathbf{q}_i(\mathbf{k}_i, \mathbf{k}_{-i}) = \mathbf{D}(\mathbf{v})\mathbf{D}^{-1}\left(\sum_{j\in\mathcal{N}} \mathbf{k}_j\right)\mathbf{k}_i.$$

Now, the distributed control problem with price anticipators can be defined as follows:

Definition 5 (*The Distributed Control Problem with Price Anticipators*)**.** The distributed control problem for price anticipators is given by

$$\mathbf{k}_i^*(\mathbf{k}_{-i}) \in \arg\max_{k_i} \{U_i(\mathbf{q}_i(\mathbf{k}_i, \mathbf{k}_{-i})) - \mathbf{1}\cdot\mathbf{k}_i : \mathbf{q}_i(\mathbf{k}_i, \mathbf{k}_{-i}) \in \mathcal{S}_i\}. \tag{33}$$

Each consumer will try to maximize her own net utility, assuming that all other consumers' expenditures are fixed and the mechanism to set the price is known. This is called the *best response strategy* that is given by a Nash equilibrium. In a Nash equilibrium no player has an incentive to deviate unilaterally of the equilibrium [26].

Definition 6 (*Nash Equilibrium for Price Anticipators*)**.** The Nash equilibrium for the distributed control problem with price anticipators is the set of expenditures $\{\mathbf{k}_i^G : i \in \mathcal{N}\}$ such that

$$U_i(\mathbf{q}_i(\mathbf{k}_i^G, \mathbf{k}_{-i}^G)) - \mathbf{1} \cdot \mathbf{k}_i^G \geq U_i(\mathbf{q}_i(\mathbf{k}_i, \mathbf{k}_{-i}^G)) - \mathbf{1} \cdot \mathbf{k}_i, \ \mathbf{q}_i(\mathbf{k}_i, \mathbf{k}_{-i}) \in \mathcal{S}_i, \ i \in \mathcal{N}. \quad (34)$$

The solution to the distributed control problem with price anticipators defines a Nash equilibrium $\mathbf{k}_i^G = \mathbf{k}_i^*(\mathbf{k}_{-i}^G)$, $i \in \mathcal{N}$, and can be characterized using the optimality conditions of problem (33). Let us consider the Lagrangian

$$\mathcal{L}_i^a = U_i(\mathbf{q}_i(\mathbf{k}_i, \mathbf{k}_{-i})) - \mathbf{1} \cdot \mathbf{k}_i + \boldsymbol{\mu}_i \cdot (\mathbf{b}_i - \mathbf{H}_i \mathbf{q}_i(\mathbf{k}_i, \mathbf{k}_{-i})) .$$

The set of vectors $\{\mathbf{k}_i^G : i \in \mathcal{N}\}$ is a Nash equilibrium for the distributed control problem with price anticipators if there exists a set $\{\boldsymbol{\mu}_i^G : i \in \mathcal{N}\}$ satisfying the following KKT conditions:

$$\mathbf{D}^{-1}(\mathbf{p}(\mathbf{k}_i^G, \mathbf{k}_{-i}^G)) \left(\mathbf{I} - \mathbf{D}^{-1}(\mathbf{v})\mathbf{D}(\mathbf{q}_i(\mathbf{k}_i^G, \mathbf{k}_{-i}^G))\right) \nabla U_i(\mathbf{q}_i(\mathbf{k}_i^G, \mathbf{k}_{-i}^G)) - \mathbf{1} - \mathbf{D}^{-1}(\mathbf{p}(\mathbf{k}_i^G, \mathbf{k}_{-i}^G)) \left(\mathbf{I} - \mathbf{D}^{-1}(\mathbf{v})\mathbf{D}(\mathbf{q}_i(\mathbf{k}_i^G, \mathbf{k}_{-i}^G))\right) \mathbf{H}^\top \mu = \mathbf{0}, \ i \in \mathcal{N}, \quad (35)$$

$$\boldsymbol{\mu}_i^G \cdot \left(\mathbf{b}_i - \mathbf{H}_i \mathbf{q}_i(\mathbf{k}_i^G, \mathbf{k}_{-i}^G)\right) = \mathbf{0}, \ i \in \mathcal{N}, \quad (36)$$

$$\mathbf{b}_i - \mathbf{H}_i \mathbf{q}_i(\mathbf{k}_i^G, \mathbf{k}_{-i}^G) \geq \mathbf{0}, \ i \in \mathcal{N}, \quad (37)$$

$$\boldsymbol{\mu}_i^G \geq \mathbf{0}, \ i \in \mathcal{N}. \quad (38)$$

Now, we will prove that a Nash equilibrium always exists. The intuition behind this claim is that the previous conditions are also the optimality conditions for an optimization program that always has a solution if the set of inequality constraints for the problem is nonempty.

Theorem 2 (*Existence of Nash Equilibrium*). *The competitive game given by* Eq. (34) *has a Nash equilibrium if the search space is nonempty.*

Proof. Let us consider the optimization program

$$\max_{\mathbf{q}_i} \sum_{i \in \mathcal{N}} \widehat{U}_i(\mathbf{q}_i) \quad (39)$$

$$\text{s.t. } \mathbf{v} - \sum_{j \in \mathcal{N}} \mathbf{q}_j = \mathbf{0}, \quad (40)$$

$$\mathbf{b}_i - \widehat{\mathbf{h}}_i(\mathbf{q}_i) \geq \mathbf{0}, i \in \mathcal{N}. \quad (41)$$

The corresponding Lagrangian is:

$$\mathcal{L}^o = \sum_{i \in \mathcal{N}} \widehat{U}_i(\mathbf{q}_i) + \mathbf{p} \cdot (\mathbf{v} - \sum_{j \in \mathcal{N}} \mathbf{q}_j) + \boldsymbol{\mu}_i \cdot \left(\mathbf{b}_i - \widehat{\mathbf{h}}_i(\mathbf{q}_i)\right) . \quad (42)$$

The KKT conditions for this optimization program are equivalent to (35)–(38) if $\widehat{U}_i(\mathbf{q}_i)$ and $\widehat{\mathbf{h}}_i(\mathbf{q}_i)$ satisfy that:

$$\nabla \widehat{U}_i(\mathbf{q}_i) = (\mathbf{I} - \mathbf{D}^{-1}(\mathbf{v})\mathbf{D}(\mathbf{q}_i)) \nabla U_i(\mathbf{q}_i), \quad (43)$$

$$\nabla \widehat{\mathbf{h}}_i(\mathbf{q}_i) = (\mathbf{I} - \mathbf{D}^{-1}(\mathbf{v})\mathbf{D}(\mathbf{q}_i)) \mathbf{H}_i^\top, \quad (44)$$

and $\nabla \widehat{\mathbf{h}}_i(\mathbf{q}_i^*) = \mathbf{H}_i \mathbf{q}_i^*$. Since the objective function (39) is continuous and the search space defined by the constraints (40) and (41) is compact, by Weierstrass theorem, the optimization program attains a maximum, and therefore a Nash equilibrium exists. □

Assuming that the search space $\mathcal{S}$ is nonempty, a Nash equilibrium exists and could be obtained using the following iterative algorithm.

Algorithm 3 (*Nash Equilibrium Solution*).

1. The control authority broadcasts the available net generation $\mathbf{v}$ and a price vector $\mathbf{p}$ to all the consumers.
2. Each consumer bids the monetary value vector $\mathbf{k}_i$.
3. The control authority computes the residual $\mathbf{r} = \mathbf{v} - \mathbf{D}^{-1}(\mathbf{p}) \sum_{i \in \mathcal{N}} \mathbf{k}_i$ and updates the price vector using a subgradient algorithm with the residual $\mathbf{r}$ as a subgradient and a right step size α_k. The control authority then broadcasts the new value of price to all the consumers.
4. Each consumer computes the aggregate monetary value of the other consumers $\sum_{j \in \mathcal{N} \setminus \{i\}} \mathbf{k}_j$ with the new price vector $\mathbf{p}$, updates her own monetary value by solving the problem (33) and bids its value vector.
5. Return to step 3 until the residual satisfies $|r| < \rho$, where $\rho > 0$ is a sufficiently small number.

In order to perform step 4 of the algorithm, each consumer needs to know the sum of the monetary values bid by the other consumers, but this is readily available because

$$\sum_{j \in \mathcal{N} \setminus \{i\}} k_j(t) = p(t)v(t) - k_i(t), \ i \in \mathcal{N}. \quad (45)$$

Now, each consumer performs an update of her own bid $\mathbf{k}_i$ by solving the problem (33). This game problem formulated in terms of the monetary values is not convex. However, it can still be solved efficiently using interior point or trust region methods with barrier functions [42,43]. These methods have been proved to globally converge to a stationary point with superlinear convergence rate. So in the worst case, the number of iterations to obtain a stationary point is $O(1/\epsilon)$. Since the subgradient algorithm is used in the outer loop by the control authority, Algorithm 3 converges to a stationary point satisfying the KKT conditions (35)–(38) in $O(1/\epsilon^2)$ iterations. The computational needs for each consumer are small because the corresponding optimization problem has a small number of variables and constraints and the intra-day market problem can converge in few minutes.

7. Loss of efficiency

The selfish behavior of agents in a non-cooperative game theoretic setting renders lower performance as compared to the optimal centralized control. Price of anarchy is a measure to quantify the loss of efficiency in using game theoretic control over centralized control. The PoA is defined as the worst-case ratio of the objective function value of a Nash equilibrium of a game and that of a centralized optimal solution [26]. This concept, (more precisely, 1 − PoA) is a worst-case estimate of loss of performance due to price anticipating selfish behavior of agents.

7.1. Price of anarchy

Theorem 3. *Let* $\{\mathbf{q}_i^C : i \in \mathcal{N}\}$ *be a solution of the centralized problem* (16) *and* $\{\mathbf{q}_i^G : i \in \mathcal{N}\}$ *a Nash equilibrium for the distributed problem with price anticipating consumers. Let F be defined by:*

$$F := \frac{\sum_{i \in \mathcal{N}} U_i(\mathbf{q}_i^G)}{\sum_{i \in \mathcal{N}} U_i(\mathbf{q}_i^C)},$$

then $F \geq 0.75$ *and the price of anarchy is 0.75.*

Proof. Since the utility functions $\{U_i : i \in \mathcal{N}\}$ are nonnegative, strictly increasing and concave, for any $\mathbf{q}_i$, then

$$0 \leq U_i(\mathbf{0}) \leq U_i(\mathbf{q}_i) - \nabla U_i(\mathbf{q}_i) \cdot \mathbf{q}_i = A(\mathbf{q}_i), \quad (46)$$

$$0 \leq U_i(\mathbf{q}_i^C) \leq U_i(\mathbf{q}_i) - \nabla U_i(\mathbf{q}_i) \cdot (\mathbf{q}_i - \mathbf{q}_i^C), \quad (47)$$

and

$$\frac{\sum_{i \in \mathcal{N}} U_i(\mathbf{q}_i)}{\sum_{i \in \mathcal{N}} U_i(\mathbf{q}_i^C)} \geq \frac{\sum_{i \in \mathcal{N}} U_i(\mathbf{q}_i)}{\sum_{i \in \mathcal{N}} U_i(\mathbf{q}_i) - \sum_{i \in \mathcal{N}} \nabla U_i(\mathbf{q}_i) \cdot (\mathbf{q}_i - \mathbf{q}_i^C)} = \frac{A(\mathbf{q}_i) + \sum_{i \in \mathcal{N}} \nabla U_i(\mathbf{q}_i) \cdot \mathbf{q}_i}{A(\mathbf{q}_i) + \sum_{i \in \mathcal{N}} \nabla U_i(\mathbf{q}_i) \cdot \mathbf{q}_i^C} \geq \frac{\sum_{i \in \mathcal{N}} \nabla U_i(\mathbf{q}_i) \cdot \mathbf{q}_i}{\sum_{i \in \mathcal{N}} \nabla U_i(\mathbf{q}_i) \cdot \mathbf{q}_i^C},$$

where the inequalities follow from (46) and (47). Now for $\mathbf{q}_i = \mathbf{q}_i^G$, defining $\mathbf{a}_i(\mathbf{q}_i^G) = \nabla_i U_i(\mathbf{q}_i^G)$, we obtain

$$\begin{aligned}\frac{\sum_{i\in\mathcal{N}} U_i(\mathbf{q}_i^G)}{\sum_{i\in\mathcal{N}} U_i(\mathbf{q}_i^C)} &\geq \frac{\sum_{i\in\mathcal{N}} \mathbf{a}_i(\mathbf{q}_i^G)\cdot\mathbf{q}_i^G}{\sum_{i\in\mathcal{N}} \mathbf{a}_i(\mathbf{q}_i^G)\cdot\mathbf{q}_i^C} \\ &= \frac{\sum_{i\in\mathcal{N}}\sum_{t\in\mathcal{T}} a_i(t) q_i^G(t)}{\sum_{i\in\mathcal{N}}\sum_{t\in\mathcal{T}} a_i(t) q_i^C(t)} \\ &\geq \frac{\sum_{i\in\mathcal{N}}\sum_{t\in\mathcal{T}} a_i(t) q_i^G(t)}{\sum_{t\in\mathcal{T}} \max_i a_i(t) \sum_{i\in\mathcal{N}} q_i^C(t)}, \end{aligned} \tag{48}$$

where $a_i(t)$ denotes the value of $\mathbf{a}_i(\mathbf{q}_i^G)$ at time $t \in \mathcal{T}$, and taking into account $\sum_{i\in\mathcal{N}} q_i^C(t) = v(t)$, a worst-case bound γ can be obtained by solving the following optimization program

$$\gamma = \min \frac{\sum_{i\in\mathcal{N}}\sum_{t\in\mathcal{T}} a_i(t) q_i^G(t)}{\sum_{t\in\mathcal{T}} \max_i a_i(t) v(t)},$$

where $\mathbf{q}_i^G$ is a Nash equilibrium that satisfies the KKT conditions (35)–(38). Here, we note that if we relax $\mathbf{q}_i^G$ to vary only in the space of $\mathbf{v} - \sum_{i\in\mathcal{N}}\mathbf{q}_i = \mathbf{0}$, by removing the constraints $\mathbf{b}_i - \mathbf{H}_i(\mathbf{q}_i) \geq \mathbf{0}$, the relaxed problem allows us to obtain a lower bound of γ. So here

$$a_i(t)\left(1 - \frac{q_i^G(t)}{v(t)}\right) = p(t),$$

for each time slot $t \in \mathcal{T}$. Consider, without loss of generality that

$$\max_i a_i(t) = a_1(t).$$

Then, the optimization problem is as follows

$$\min_{q_i^G(t)} \sum_{t\in\mathcal{T}} \frac{a_1(t)}{Q}\left(q_1^G(t) + \sum_{i=2}^{n} \bar{a}_i(t) q_i^G(t)\right) \tag{49}$$

$$\text{s.t } \bar{a}_i(t)\left(1 - \frac{q_i^G(t)}{v(t)}\right) = \left(1 - \frac{q_1^G(t)}{v(t)}\right), \tag{50}$$

$$\sum_{i\in\mathcal{N}} q_i^G(t) = v(t), \tag{51}$$

$$0 \leq \bar{a}_i(t) \leq 1, \tag{52}$$

where $\bar{a}_i(t) = \frac{a_i(t)}{a_1(t)}$ and $Q = \sum_{t\in\mathcal{T}} a_1(t) v(t)$. From (50),

$$\bar{a}_i(t) = \frac{v(t) - q_1^G(t)}{v(t) - q_i^G(t)}.$$

The optimization problem can be solved in two stages. First, we fix $q_1^G(t)$ and optimize over $\{q_i^G(t) : i \in \mathcal{N} \setminus \{1\}\}$. Second, we compute the optimal value of $q_1^G(t)$. The first stage problem is:

$$\min_{\substack{q_i^G(t)\\ i\in\mathcal{N}\setminus\{1\}}} \sum_{t\in\mathcal{T}} \frac{\alpha_1(t)}{Q}\left(q_1^G(t) + \sum_{i=2}^{n} \frac{v(t) - q_1^G(t)}{v(t) - q_i^G(t)} q_i^G(t)\right)$$

$$\text{s.t. } \sum_{i=2}^{N} q_i^G(t) = v(t) - q_1^G(t),$$

$$0 \leq q_i^G(t) \leq q_1^G(t),\ i \in \mathcal{N} \setminus \{1\}.$$

For feasibility of the solution, we need to have $N q_1^G(t) \geq v(t)$. Now from symmetry, the optimal solution

$$q_i^G(t) = \frac{v(t) - q_1^G(t)}{N-1}, \quad i \neq 1, i \in \mathcal{N}.$$

After substituting the values of $q_i^G(t)$ for all $i \neq 1$, we have the following problem,

$$\min_{q_1^G(t)} \sum_{t\in\mathcal{T}} \frac{a_1(t)}{Q}\left(q_1^G(t) + \left(v(t) - q_1^G(t)\right)^2 \left(v(t) - \frac{v(t) - q_1^G(t)}{N-1}\right)^{-1}\right)$$

$$\text{s.t. } \frac{v(t)}{N} \leq q_1^G(t) \leq v(t).$$

The objective function of the above optimization problem is decreasing in N for every value of $q_1^G(t)$. So for $N \to \infty$, the worst case efficiency is obtained by solving the following problem

$$\min_{q_1^G(t)} \sum_{t\in\mathcal{T}} \frac{a_1(t)}{Q}\left(q_1^G(t) + (v(t) - q_1^G(t))^2 v(t)\right)$$

$$\text{s.t. } 0 \leq q_1^G(t) \leq v(t).$$

The solution to this problem is at $q_1^G(t) = 0.5v(t)$ which yields a worst case efficiency of 0.75. The bound is tight when the number of consumers increases to infinity and the inequality consumption constraints are not binding. □

The worst-case loss of efficiency corresponds to the case where one agent consumes half of the total power consumed by the other agents at each time slot. Note that this theoretical worst case could only be attained under a particular setting and when the load constraints are inactive. So the efficiency of the game is higher in general and it can be improved further.

7.2. Efficiency improvement

We are interested in developing strategies to improve efficiency. Clearly, the market power of a consumer plays a key role in deciding the efficiency of the game.

The following corollaries show two different ways to improve efficiency.

Corollary 1. *If all the consumers have same utility function,* i.e., $U_i = U$, *there is no efficiency loss at Nash equilibrium solution,* i.e. *PoA is 1.*

Proof. Since utility functions of all the consumers are same, using the inequality (48) we can write

$$\frac{\sum_{i\in\mathcal{N}} U_i(\mathbf{q}_i^G)}{\sum_{i\in\mathcal{N}} U_i(\mathbf{q}_i^C)} \geq \frac{\sum_{i\in\mathcal{N}}\sum_{t\in\mathcal{T}} a(t) q_i^G(t)}{\sum_{i\in\mathcal{N}}\sum_{t\in\mathcal{T}} a(t) q_i^C(t)},$$

where $a_i(t) = a(t)$ for all i. Now as $\sum_{i\in\mathcal{N}} q_i^C(t) = \sum_{i\in\mathcal{N}} q_i^G(t) = v(t)$, there is no efficiency loss at Nash equilibrium. □

Efficiency can be improved by recruiting consumers with similar utility functions, or by classifying them into groups of similar utility and designing a program for each group. So the distributed method will have better efficiency if consumers can share their utility functions with the central control authority. Another option is to reduce individual market power by increasing the number of consumers.

Corollary 2. *Suppose* $\mathbf{q}_i = \mathbf{0}$ *for all* $i \in \mathcal{N}$ *belongs to the set of load operating constraints, then the PoA approaches 1 as the number* N *of flexible consumers goes to infinity.*

Proof. From concavity of the utility function

$$\nabla U_i(\mathbf{q}_i) \leq \lim_{N\to\infty} \nabla U_i(\mathbf{q}_i) = \nabla U_i(\mathbf{0}).$$

Now, using the lower bound of the efficiency

$$\frac{\sum_{i\in\mathcal{N}} U_i(\mathbf{q}_i^G)}{\sum_{i\in\mathcal{N}} U_i(\mathbf{q}_i^C)} \geq \frac{\sum_{t\in\mathcal{T}} p^G(t) v(t)}{\sum_{t\in\mathcal{T}} \max_i \nabla_t U_i(\mathbf{q}_i^G) v(t)}.$$

Relaxing the inequality conditions in order to obtain lower bound on PoA, the Nash equilibrium condition of the game,

$$\nabla U_i(\mathbf{q}_i^G) = (\mathbf{I} - \mathbf{D}^{-1}(\mathbf{v})\mathbf{D}(\mathbf{q}_i^G))^{-1}\mathbf{p}^G,$$

when N approaches infinity converges to

$$\lim_{N\to\infty} \nabla_t U_i(\mathbf{q}_i^G) = p(t),$$

and we conclude

$$\lim_{N\to\infty} \frac{\sum_{i\in\mathcal{N}} U_i(\mathbf{q}_i^G)}{\sum_{i\in\mathcal{N}} U_i(\mathbf{q}_i^C)} = 1. \quad \square$$

This second result is also very intuitive. Since the available supply of energy is limited, as the number of flexible consumers increases, the consumptions assigned using a proportional allocation mechanism tend to equalize for every consumer and therefore, the Nash equilibrium solution converges to the competitive equilibrium and the efficiency increases to its maximum value.

8. Case study

We illustrate the paper results using a simple example with $N = 5$ consumers and $T = 8$ time slots. The utilities and constraints for each consumer are shown in Table 1, corresponding to a combination of different type of loads: interruptible, deferrable and thermostatically controlled. The parameters H and b of the 5th consumer corresponds to an air conditioning load with thermal parameters $\alpha = 0.1$, $\beta = 1$, inside reference temperature $T^{\text{conf}} = 22$, output temperatures $T^{\text{out}} = \begin{bmatrix}29 & 30 & 30 & 29 & 28 & 27 & 26 & 25\end{bmatrix}^\top$ and initial inside temperature $T^{\text{in}}(0) = 22$. The temperatures are expressed in Celsius degrees. The available power for flexible loads is $\mathbf{v} = \begin{bmatrix}1.6 & 1.8 & 2.0 & 2.4 & 3.0 & 2.6 & 2.2 & 2.8\end{bmatrix}$ in the same units of power as the consumptions. The assigned power consumptions for the cases of price takers and price anticipators are shown in Tables 2 and 3, respectively. In these tables, also the aggregated consumption for the eight time slots and the price vector are shown. The Algorithms 2 and 3 have been implemented using functions of the Optimization Toolbox in MATLAB 2016a [44] and executed in a laptop with CPU Intel Core i7-2677M at 1.80 GHz with 8 Gb of RAM under Ubuntu 16.04.2. The absolute accuracy of the residual in both cases is $\rho = 10^{-4}$. The number of iterations and the mean and standard deviation of the CPU time (in seconds) per iteration and consumer are shown in Table 4.

The price takers' solution maximizes the system welfare, and the price of anarchy for the Nash equilibrium for the case of price anticipators is 99.92%. Note that the bound of 75% of the price of anarchy is a worst case result. This case study shows that in practical cases this bound can be quite conservative, and the efficiency of the Nash equilibrium for the price anticipators under the proportional allocation rule is very high.

If we assume that all consumers are equivalent, *i.e.* their loads and utility functions are the same, then as was proved in Corollary 1, the Nash equilibrium is equivalent to the competitive equilibrium and there is no loss of efficiency. Assuming that all consumers are equivalent to consumer 3, the power consumptions and prices are shown in Table 5.

Table 1
Problem definition.

i	Utility	Constraints
1	$\sum_{t\in\mathcal{T}} 4q_i(t)(1-q_i(t))$	$0.25 \le q_i(t) \le 1.00$
2	$\sum_{t\in\mathcal{T}} (2-q_i(t))q_i(t)$	$0.25 \le q_i(t) \le 1.00$
3	$\sum_{t\in\mathcal{T}} (1-e^{-6q_i(t)})$	$0.00 \le q_i(t) \le 1.00$
		$1.00 \le \sum_{t\in\mathcal{T}} q_i(t) \le 3.00$
4	$\sum_{t\in\mathcal{T}} 0.5(1-e^{-6q_i(t)})$	$0.00 \le q_i(t) \le 1.00$
		$1.00 \le \sum_{t\in\mathcal{T}} q_i(t) \le 3.00$
5	$10(1-0.5\|b-Hx\|^2)$	$0.00 \le q_i(t) \le 1.00$
		$5.00 \le \sum_{t\in\mathcal{T}} q_i(t) \le 8.00$

Table 2
Price takers.

t	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$	$\mathbf{q}_4$	$\mathbf{q}_5$	$\mathbf{p}$
1	0.3095	0.2500	0.2240	0.1129	0.7035	1.5238
2	0.3264	0.3042	0.2394	0.1284	0.8015	1.3888
3	0.3577	0.4294	0.2717	0.1615	0.7796	1.1381
4	0.4181	0.6724	0.3607	0.2537	0.6951	0.6548
5	0.4734	0.8946	0.5334	0.4411	0.6574	0.2126
6	0.4574	0.8310	0.4636	0.3625	0.4855	0.3408
7	0.4304	0.7220	0.3866	0.2807	0.3803	0.5567
8	0.4710	0.8846	0.5204	0.4267	0.4971	0.2318
$\sum_{t\in\mathcal{T}}$	3.2441	4.9882	3.0000	2.1676	5.0000	

Table 3
Price anticipators.

t	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$	$\mathbf{q}_4$	$\mathbf{q}_5$	$\mathbf{p}$
1	0.3071	0.2572	0.2249	0.1320	0.6787	1.2467
2	0.3273	0.3148	0.2420	0.1483	0.7676	1.1306
3	0.3597	0.4183	0.2754	0.1811	0.7654	0.9202
4	0.4178	0.6314	0.3585	0.2653	0.7271	0.5432
5	0.4744	0.8775	0.5287	0.4480	0.6714	0.1736
6	0.4581	0.8005	0.4586	0.3718	0.5111	0.2763
7	0.4354	0.6967	0.3954	0.3050	0.3675	0.4145
8	0.4721	0.8652	0.5166	0.4348	0.5112	0.1866
$\sum_{t\in\mathcal{T}}$	3.2519	4.8616	3.0000	2.2863	5.0000	

9. Conclusion

In this paper, we have investigated supply–demand balancing by controlling flexible power consumption in an intra-day time horizon in case of large scale renewable integration. We modeled the flexible consumers with a general utility function which is not necessarily separable with respect to time intervals and a set of linear constraints considering the flexibility associated with their loads. We considered a non-profit central control authority that balances the supply–demand and maximizes the system welfare by using a price signal inspired by Kelly's proportional allocation mechanism and we proved that if consumers behave as price takers, the overall system achieves a competitive equilibrium. This method can be applied to a power system with high renewable penetration in an intra-day time horizon and it will help to mitigate supply–demand imbalance in real time.

We next investigated the more interesting case of price anticipating consumers where system is modeled in as a non-cooperative Cournot game that achieves a Nash equilibrium. The selfish behavior of the consumers may lead to loss of efficiency, but we were able to achieve a worst case maximum efficiency loss of 25%. This loss of efficiency depends on the utility function of the consumers and number of consumers. We showed that if utility functions of all the consumers are same, then there is no efficiency loss. Also, we showed that with consumers having different utility functions, if the number of consumers tend to increase to infinity, the efficiency loss approaches zero.

Future extensions of out work include dynamic games to model the intermittency effects of renewable sources and the study of price based DR programs with an utility company having own profit interests. In the second case, Stackelberg model could be applied where the consumers responds to the price signal decided first by a utility company.

Table 4
Performance of the algorithms.

Algorithm	# iterations	CPU time (s) / iteration with consumers
Competitive equilibrium	25	0.0344 ± 0.0267
Nash equilibrium	29	0.1049 ± 0.0219

Table 5
Price takers and price anticipators with same utility functions.

t	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$	$\mathbf{q}_4$	$\mathbf{q}_5$	$\mathbf{p}$
1	0.3200	0.3200	0.3200	0.3200	0.3200	1.3600
2	0.3600	0.3600	0.3600	0.3600	0.3600	1.2800
3	0.4000	0.4000	0.4000	0.4000	0.4000	1.2000
4	0.4800	0.4800	0.4800	0.4800	0.4800	1.0400
5	0.6000	0.6000	0.6000	0.6000	0.6000	0.8000
6	0.5200	0.5200	0.5200	0.5200	0.5200	0.9600
7	0.4400	0.4400	0.4400	0.4400	0.4400	1.1200
8	0.5600	0.5600	0.5600	0.5600	0.5600	0.8800
$\sum_{t\in\mathcal{T}}$	3.6800	3.6800	3.6800	3.6800	3.6800	

References

[1] NERC, accomodation of high levels of variable generation, Tech. Report, NERC, April 2009.
[2] E. Bitar, P.P. Khargonekar, K. Poolla, Systems and control opportunities in the integration of renewable energy into the smart grid, in: Proceedings of the 18th IFAC World Congress, Milano (Italy), 2011, pp. 4927–4932.
[3] P. Denholm, W. Short, An evaluation of utility system impacts and benefits of optimally dispatched plug-in hybrid electric vehicles, Technical Report NREL/TP-620-40293, National Renewable Energy Laboratory, Amherst, MA, 2006.
[4] A.J. Roscoe, G. Ault, Supporting high penetrations of renewable generation via implementation of real-time electricity pricing and demand response, IET Renew. Power Gen. 4 (4) (2010) 369–382.
[5] J. Aghaei, M.-I. Alizadeh, Demand response in smart electricity grids equipped with renewable energy sources: A review, Elsevier Renewable and Sustainable Energy Reviews 18 (2013) 64–72.
[6] F. Borggrefe, K. Neuhoff, Balancing and intraday market design: Options for Wind Integration, Technical Report, DIW Berlin, German Institute for Economic Research, 2011.
[7] J.M. Morales, A.J. Conejo, H. Madsen, P. Pinson, M. Zugno, Integrating renewables in electricity markets: operational problems, Springer Science & Business Media, 2013.
[8] M.H. Albadi, E. El-Saadany, A summary of demand response in electricity markets, Electr. Pow. Syst. Res. 78 (11) (2008) 1989–1996.
[9] A. Faruqui, S. Sergici, Household response to dynamic pricing of electricity: A survey of 15 experiments, J. Regul. Econ. 38 (2) (2010) 193–225.
[10] S. Rehman, G. Shrestha, An investigation into the impact of electric vehicle load on the electric utility distribution system, IEEE Trans. Power Del. 8 (2) (1993) 591–597.
[11] D. Callaway, I. Hiskens, Acheiving controllability of electric loads, Proc. IEEE 99 (1) (2011) 184–199.
[12] H. Hao, A. Kowli, Y. Lin, P. Barooah, S. Meyn, Ancillary service for the grid via control of commercial building HVAC systems, in: American Control Conference (ACC), 2013, pp. 467–472.
[13] N.H. Tran, C.T. Do, S. Ren, Z. Han, C.S. Hong, Incentive mechanisms for economic and emergency demand responses of colocation datacenters, IEEE J. Sel. Area Commun. 33 (12) (2015) 2892–2905.
[14] N.H. Tran, D.H. Tran, S. Ren, Z. Han, E.N. Huh, C.S. Hong, How Geo-distributed data centers do demand response: A game-theoretic approach, IEEE Trans. Smart Grid 7 (2) (2016) 937–947.
[15] N. Li, L. Chen, S.H. Low, Optimal demand response based on utility maximization in power networks, in: Proceedings of IEEE Power Engineering Society General Meeting, San Diego, CA, 2011, pp. 1–8.
[16] W. Shi, N. Li, X. Xie, C. Chu, R. Gadh, Optimal residential demand response in distribution networks, IEEE J. Sel. Areas Commun. 32 (7) (2014) 1441–1450.
[17] F.P. Kelly, Charging and rate control for elastic traffic, European Trans. Telecommun. 8 (1) (1997) 33–37.
[18] F.P. Kelly, A.K. Maulloo, D.K.H. Tan, Rate control in communication networks: Shadow prices, proportional prices and stability, J. Oper. Res. Soc. 49 (3) (1998) 237–252.
[19] Z. Fan, Distributed charging of PHEVs in a smart grid, in: Proceedings of the IEEE International Conference on Smart Grid Communications, Brussels, 2011, pp. 255–260.
[20] M. Vasirani, S. Ossowski, A proportional share allocation mechanism for coordination of plug-in electric vehicle charging, Eng. Appl. Artif. Intell. 26 (3) (2013) 1185–1197.
[21] Y. Li, The case analysis of the scandal of Enron, Int. J. Bus. Manag. 5 (10) (2010) P37.
[22] W. Saad, A.L. Glass, N.B. Mandayam, H.V. Poor, Toward a consumer-centric grid: A behavioral perspective, Proc. IEEE 104 (4) (2016) 865–882.
[23] Y. Wang, W. Saad, N.B. Mandayam, H.V. Poor, Load shifting in the smart grid: To participate or not? IEEE Trans. Smart Grid 7 (6) (2016) 2604–2614.
[24] R. Johari, Efficiency loss in market mechanisms for resource allocation, Ph.D. Thesis, Massachusetts Institute of Technology. Department of Electrical Engineering and Computer Science, 2004.
[25] H. Moulin, The price of anarchy of serial, average and incremental cost sharing, Econ. Theory 36 (3) (2008) 379–405.
[26] T. Roughgarden, Intrinsic robustness of the price of anarchy, Commun. ACM 55 (7) (2012) 116–123.
[27] T. Roughgarden, Routing games, in: N. Nisan, T. Roughgarden, E. Tardos, V.V. Vazirani (Eds.), Algorithmic Game Theory, Cambridge University Press, Avenue of the Americas, NY, 2007, pp. 459–484.
[28] R. Johari, The price of anarchy and the design of scalable resource allocation mechanisms, in: N. Nisan, T. Roughgarden, E. Tardos, V.V. Vazirani (Eds.), Algorithmic Game Theory, Cambridge University Press, Avenue of the Americas, NY, 2007, pp. 543–568.
[29] Z. Ma, D. Callaway, I. Hiskens, Decentralized charging control of large populations of plug-in electric vehicles, IEEE Trans. Control Syst. Technol. 21 (1) (2013) 67–78.
[30] C. Wu, H. Mohsenian-Rad, J. Huang, Wind power integration via aggregator-consumer coordination: A game theoretic approach, in: Proceedings of IEEE PES Innovative Smart Grid Technologies, Washington, DC, 2012, pp. 1–6.
[31] Q. Huang, M. Roozbelani, M. Dahleh, Efficiency-risk tradeoffs in dynamic oligopoly markets, in: Proceedings of the 51st IEEE Conference on Decision and Control, Maui, HI, 2012, pp. 2388–2394.
[32] A. Mohsenian-Rad, V. Wong, J. Jatskevich, R. Schober, A.L. Garcia, Autonomous demand-side management based on game-theoritic energy consumption scheduling for the future smart grid, IEEE Trans. on Smart Grid 1 (3) (2010) 320–331.
[33] P. Chakraborty, P.P. Khargonekar, A demand response game and its robust price of anarchy, in: Proceedings of the 2014 IEEE Control Conference on Smart Grid Communications (Smart Grid Comm), Venice, Italy, 2014, pp. 644–649.
[34] P. Chakraborty, P.P. Khargonekar, Flexible loads and renewable integration: Distributed control and price of anarchy, in: Proceedings of the IEEE 52nd Annual Conference on Decision and Control (CDC), Firenze, Italy, 2013, pp. 2306–2312.
[35] S. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press, Cambridge, Uk, 2009.
[36] A. Nedic, A. Ozdaglar, Distributed subgradient methods for multi-agent optimization, IEEE Trans. Autom. Control 54 (1) (2009) 48–61.
[37] D.P. Bertsekas, J.N. Tsitsklis, Parallel and Distributed Computation, Prentice Hall, Englewood Cliffs, NJ, 1989.
[38] Y. Censor, S.A. Zenios, Parallel Optimization: Theory, Algorithms, and Applications, Oxford University Press, 1997.
[39] A. Nedic, D.P. Bertsekas, Incremental subgradient methods for nondifferentiable optimization, SIAM J. Optim. 12 (1) (2001) 109–138.
[40] Y. Nesterov, in: Introductory lectures on convex optimization: A basic course, Vol. 87, Springer Science & Business Media, 2004.
[41] H. Yabe, H. Yamashita, Q-superlinear convergence of primal-dual interior point quasi-Newton methods for constrained optimization, J. Oper. Res. Soc. Japan 40 (3) (1997) 415–436.
[42] A. Forsgren, P.E. Gill, Primal-dual interior methods for nonconvex nonlinear programming, SIAM J. Optim. 8 (4) (1998) 1132–1152.
[43] R.H. Byrd, J.C. Gilbert, J. Nocedal, A trust region method based on interior point techniques for nonlinear programming, Math. Prog. 89 (1) (2000) 149–185.
[44] T. Mathworks, Matlab optimization toolbox user's guide, 2016.